Construction of Ricci-type connections by reduction and induction.

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Abstract

Given the Euclidean space $\mathbb{R}^{2n+2}$ endowed with a constant symplectic structure and the standard flat connection, and given a polynomial of degree 2 on that space, Baguis and Cahen [1] have defined a reduction procedure which yields a symplectic manifold endowed with a Ricci-type connection. We observe that any symplectic manifold (M, ω) of dimension $2n$ $(n \geq 2)$ endowed with a symplectic connection of Ricci type is locally given by a local version of such a reduction.

We also consider the reverse of this reduction procedure, an induction procedure: we construct globally on a symplectic manifold endowed with a connection of Ricci-type (M, ω, ∇) a circle or a line bundle which embeds in a flat symplectic manifold (P, μ, ∇^1) as the zero set of a function whose third covariant derivative vanishes, in such a way that (M, ω, ∇) is obtained by reduction from (P, μ, ∇^1).

We further develop the particular case of symmetric symplectic manifolds with Ricci-type connections.

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1 Introduction

A symplectic connection ∇ on a symplectic manifold (M, ω) of dimension $2n$ is a linear connection which is torsion free and for which ω is parallel. The space of symplectic connections on (M, ω), $\mathscr{E}(M, \omega)$ is infinite dimensional.

Selecting some particular class of connections by curvature conditions has, a priori, two interests. The "moduli space" of such particular connections may be finite dimensional; also, on some compact symplectic manifolds which do admit a connection of the chosen class, this connection may be "rigid".

In this paper, we describe completely the local behaviour of symplectic connections of Ricci-type (see definition below) and give some global description of simply connected symplectic manifolds admitting a connection of Ricci-type.

We denote by R the curvature of ∇ and by $\underline{R}$ the symplectic curvature tensor

$$\underline{R}(X, Y, Z, T) := \omega(R(X, Y)Z, T).$$

For any point $x \in M$, we have the symmetry properties

(i) $\underline{R}_x(X, Y, Z, T) = -\underline{R}_x(Y, X, Z, T)$

(ii) $\underline{R}_x(X, Y, Z, T) = \underline{R}_x(X, Y, T, Z)$

(iii) $\displaystyle \underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\bigoplus} \underline{R}_x(X, Y, Z, T) = 0.$

From (i) and (ii), $\underline{R}_x \in \Lambda^2 T_x^* M \otimes \odot^2 T_x^* M$, where $\odot^k V$ is the symmetrized k-tensor product of the vector space V.

We denote by r the Ricci tensor of the connection ∇ (i.e. $r_x(X, Y) = \mathrm{tr}\ [Z \to R_x(X, Z)Y]$, where X, Y, Z are in T_xM); this tensor r is symmetric. We denote by ρ the corresponding endomorphism of the tangent bundle:

$$\omega(X, \rho Y) \underset{\text{def}}{=} r(X, Y)$$

so that ρ_x belongs to the symplectic algebra $sp(T_xM, \omega_x)$; in particular $\mathrm{tr}\ \rho = 0$.

The space $\mathscr{R}_x$ of symplectic curvature tensors at x is

$$\mathscr{R}_x = \ker a \subset \Lambda^2 T_x^* M \otimes \odot^2 T_x^* M$$

where a is the skewsymmetrisation map $a : \Lambda^p T_x^* M \otimes \odot^q T_x^* M \to \Lambda^{p+1} T_x^* M \otimes \odot^{q-1} T_x^* M$

$$a(u_1 \wedge \ldots \wedge u_p \otimes v_1 \ldots v_q) := \sum_{i=1}^{q} u_1 \wedge \ldots \wedge u_p \wedge v_i \otimes v_1 \ldots \hat{v}_i \ldots v_q.$$

The group $Sp(T_xM, \omega_x)$ acts on $\mathscr{R}_x$. Under this action the space $\mathscr{R}_x$, in dimension $2n \geq 4$, decomposes into two irreducible subspaces [6]:

$$\mathscr{R}_x = \mathscr{E}_x \oplus \mathscr{W}_x$$

and the decomposition of the curvature tensor $\underline{R}_x$ into its $\mathscr{E}_x$ component (denoted E_x) and its $\mathscr{W}_x$ component (denoted W_x) , $\underline{R}_x = E_x + W_x$, is given by

$$\begin{aligned} E_x(X,Y,Z,T) \quad = \quad & -\frac{1}{2(n+1)}\big[2\omega_x(X,Y)r_x(Z,T) + \omega_x(X,Z)r_x(Y,T) \\ & +\omega_x(X,T)r_x(Y,Z) - \omega_x(Y,Z)r_x(X,T) - \omega_x(Y,T)r_x(X,Z)\big] \end{aligned}$$

A connection ∇ is said to be **of Ricci-type** if, at each point x, $W_x = 0$.(Let us mention that such connections were called reducible by Vaisman in [6]). In dimension 2 ($n = 1$), the space $\mathscr{W}$ vanishes identically; so we shall assume in what follows that the manifold has dimension $m = 2n > 2$.

Let us first recall two interesting features of such connections.

- When a symplectic connection is of Ricci-type, is satisfies the equations:

$$\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\!\bigoplus} (\nabla_X r)(Y,Z) = 0.$$

Those are the Euler-Lagrange equations of any natural variational principle whose Lagrangian is a second degree invariant polynomial in the curvature (r^2 or R^2). Connections which are solutions of those equations are called preferred ; they are completely described in dimension 2.

- The condition to be of Ricci-type is the condition on a symplectic connection ∇ to have an integrable almost complex structure J^∇ on the twistor space over M which is the bundle of all compatible almost complex structures on M ([2]).

In this paper, we show that any symplectic manifold (M, ω) of dimension $2n$ ($n \geq 2$) admitting a symplectic connection of Ricci type has a local model given by a reduction procedure (as introduced by Baguis and Cahen in [1]) from the Euclidean space $\mathbb{R}^{2n+2}$ endowed with a constant symplectic structure and the standard flat connection.

We also consider the reverse of this reduction procedure, an induction procedure: we construct globally on a simply connected symplectic manifold endowed with a connection of Ricci-type (M, ω, ∇) a circle or a line bundle N which embeds in a flat symplectic manifold (P, μ, ∇^1) as the zero set of a function whose third covariant derivative vanishes, in such a way that (M, ω, ∇) is obtained by reduction from (P, μ, ∇^1).

We finally describe completely the symmetric symplectic manifolds whose canonical connection is of Ricci-type. Those were already studied in [4] in collaboration with John Rawnsley.

2 Some properties of the curvature of a Ricci-type connection

Let (M, ω) be a smooth symplectic manifold of dim $2n$ ($n \geq 2$) and let ∇ be a smooth Ricci-type symplectic connection. The following results follow directly from the definition (and Bianchi's second identity).

Lemma 2.1 *[3] The curvature endomorphism reads*

$$R(X,Y) = -\frac{1}{2(n+1)}[-2\omega(X,Y)\rho - \rho Y \otimes \underline{X} + \rho X \otimes \underline{Y} - X \otimes \underline{\rho Y} + Y \otimes \underline{\rho X}] \quad (1)$$

where $\underline{X}$ denotes the 1-form $i(X)\omega$ $(for X$ a vector field on $M)$ and where, as before, ρ is the endomorphism associated to the Ricci tensor [$r(U,V) = \omega(U, \rho V)$].

Furthermore:

(i) there exists a vector field u such that

$$\nabla_X \rho = -\frac{1}{2n+1}[X \otimes \underline{u} + u \otimes \underline{X}]; \quad (2)$$

(ii) there exists a function f such that

$$\nabla_X u = -\frac{2n+1}{2(n+1)}\rho^2 X + fX; \quad (3)$$

(iii) there exists a real number K such that

$$tr\rho^2 + \frac{4(n+1)}{2n+1}f = K. \quad (4)$$

3 Construction by reduction of manifolds with Ricci type connections

Let A be a nonzero element in the symplectic Lie algebra $sp(\mathbb{R}^{2n+2}, \Omega')$ where Ω' is the standard symplectic structure on $\mathbb{R}^{2n+2}$. Let Σ_A be the closed hypersurface $\Sigma_A \subset \mathbb{R}^{2n+2}$ with equation :

$$\Omega'(x, Ax) = 1; \quad (5)$$

in order for Σ_A to be non empty we replace, if necessary, A, by $-A$.

Let $\dot{\nabla}$ be the standard flat symplectic affine connection on $\mathbb{R}^{2n+2}$. If X, Y are vector fields tangent to Σ_A define:

$$(\nabla^{\Sigma_A}_X Y)(x) = (\dot{\nabla}_X Y)(x) - \Omega'(AX, Y)x; \quad (6)$$

this is a torsion free linear connection on Σ_A.

The vector field Ax is an affine vector field for this connection; it is clearly complete and we denote by ϕ_t the 1-parametric group of diffeomorphisms of Σ_A generated by this vector field; clearly this flow is given by the restriction to Σ_A of the action of $\exp tA$ on $\mathbb{R}^{2n+2}$.

Since the vector field Ax is nowhere 0 on Σ_A, for any $x_0 \in \Sigma_A$, there exists :

-a neighborhood $U_{x_0}(\subset \Sigma_A)$,

-a ball $D \subset \mathbb{R}^{2n}$ of radius r_0, centered at the origin,

-a real interval $I = (-\epsilon, \epsilon)$

-and a diffeomorphism

$$\chi : D \times I \to U_{x_0} \tag{7}$$

such that $\chi(0,0) = x_0$ and $\chi(y,t) = \phi_t(\chi(y,0))$. We shall denote

$$\pi : U_{x_0} \to D \quad \pi = p_1 \otimes \chi^{-1}.$$

If we view Σ_A as a constraint manifold in $\mathbb{R}^{2n+2}$, D is a local version of the Marsden-Weinstein reduction of Σ_A around the point x_0.

If $x \in \Sigma_A$, $T_x\Sigma_A = \rangle Ax\langle^{\perp}$, where $\rangle v_1, \ldots, v_p\langle$ denotes the subspace spanned by $v_1, \ldots, v_p$ and $^{\perp}$ denotes the orthogonal relative to Ω'; let $\mathscr{H}_x(\subset T_x\Sigma_A) = \rangle x, Ax\langle^{\perp}$; then

$$T_x\mathbb{R}^{2n+2} = (\mathscr{H}_x \oplus \mathbb{R}Ax) \oplus \mathbb{R}x$$

and π_{*x} defines an isomorphism between $\mathscr{H}_x$ and the tangent space T_yD for $y = \pi(x)$. A vector belonging to $\mathscr{H}_x$ will be called horizontal.

A symplectic form on D, ω, is defined by

$$\omega_y(X,Y) = \Omega'_x(\bar{X}, \bar{Y}) \qquad y = \pi(x) \tag{8}$$

where $\bar{X}$ (resp. $\bar{Y}$) denotes the horizontal lift of X (resp. Y). A symplectic connection ∇ on D is defined by

$$\overline{\nabla_X Y}(x) = \nabla^{\Sigma_A}_{\bar{X}}\bar{Y}(x) + \Omega'(\bar{X},\bar{Y})Ax \tag{9}$$

Proposition 3.1 *[1] The manifold (D,ω) is a symplectic manifold and ∇ is a symplectic connection of Ricci-type.*

Furthermore, a direct computation shows that the corresponding ρ, u and f are given

by:

$$\begin{aligned}\overline{\rho X}(x) &= -2(n+1)\overline{A_x}\bar{X} && (10)\\ \bar{u}(x) &= -2(n+1)(2n+1)\overline{A_x^2}x && (11)\\ (\pi^* f)(x) &= 2(n+1)(2n+1)\Omega'(A^2x, Ax) && (12)\end{aligned}$$

where$\overline{A^k_x}$ *is the map induced by* A^k *with values in* $\mathscr{H}_x$*:*

$$\overline{A^k_x}(X) = A^kX + \Omega'(A^kX, x)Ax - \Omega'(A^kX, Ax)x$$

4 Local models for symplectic connections of Ricci-type

The properties of a symplectic connection of Ricci-type, as stated in Lemma 2.1, imply in particular that

-the curvature tensor is determined by ρ;

-its covariant derivative is determined by u;

-its second covariant derivative is determined by ρ and f, hence by ρ and K with K a constant;

-the 3rd covariant derivative of the curvature is determined by u, ρ, K and similarly for all orders.

Hence

Corollary 4.1 *Let* (M, ω) *be a smooth symplectic manifold of dimension* $2n$ $(n \geq 2)$ *and let* ∇ *be a smooth Ricci-type connection. Let* $p_0 \in M$*; then the curvature* R_{p_0} *and its covariant derivatives* $(\nabla^k R)_{p_0}$ *(for all k) are determined by* (ρ_{x_0}, u_{x_0}, K)*.*

Corollary 4.2 *Let* (M, ω, ∇) *(resp.* (M', ω', ∇')*) be two real analytic symplectic manifolds of the same dimension* $2n$ $(n \geq 2)$ *each of them endowed with a symplectic connection of Ricci-type.*

Assume that there exists a linear map $b : T_{x_0}M \to T_{x_0'}M'$ *such that (i)* $b^*\omega'_{x_0'} = \omega_{x_0}$ *(ii)* $bu_{x_0} = u'_{x_0'}$ *(iii)* $b \circ \rho_{x_0} \circ b^{-1} = \rho'_{x_0'}$*. Assume further that* $K = K'$*.*

Then the manifolds are locally affinely symplectically isomorphic, i. e. there exists a normal neighborhood of x_0 *(resp.* x_0'*)* U_{x_0} *(resp.* $U'_{x_0'}$*) and a symplectic affine diffeomorphism* $\varphi : (U_{x_0}, \omega, \nabla) \to (U'_{x_0'}, \omega', \nabla')$ *such that* $\varphi(x_0) = x_0'$ *and* $\varphi_{*x_0} = b$*.*

This follows from classical results, see for instance theorem 7.2 and corollary 7.3 in Kobayashi-Nomizu volume 1 [5].

Consider now (M, ω, ∇) a real analytic symplectic manifold of dimension $2n$ ($n \geq 2$) endowed with an analytic Ricci-type symplectic connection; denote as before by u, ρ, f and K the associated quantities (see lemma 2.1).

Let p_0 be a point in M and choose ξ_0 a symplectic frame of $T_{p_0}M$, i.e. a linear symplectic isomorphism $\xi_0 : (\mathbb{R}^{2n}, \Omega) \to (T_{p_0}, \omega_{p_0})$, where Ω is the standard symplectic form on $\mathbb{R}^{2n}$.

Denote by $\tilde{u}(\xi_0)$ the element of $\mathbb{R}^{2n}$ corresponding to $u(p_0)$, i.e.

$$\tilde{u}(\xi_0) = (\xi_0)^{-1}\ u(p_0)$$

and by $\tilde{\rho}(\xi_0)$ the element of $sp(\mathbb{R}^{2n}, \Omega)$ corresponding to $\rho(p_0)$, i.e.

$$\tilde{\rho}(\xi) = (\xi_0)^{-1}\ \rho(p_0)\ \xi_0.$$

Define an element A of $sp((\mathbb{R}^{2n+2}, \Omega')$ as:

$$A = \begin{pmatrix} 0 & \dfrac{f(p_0)}{2(n+1)(2n+1)} & \dfrac{-\underline{\tilde{u}(\xi_0)}}{2(n+1)(2n+1)} \\ 1 & 0 & 0 \\ 0 & \dfrac{-\tilde{u}(\xi_0)}{2(n+1)(2n+1)} & \dfrac{-\tilde{\rho}(\xi_0)}{2(n+1)} \end{pmatrix}$$

where $\underline{\tilde{u}(\xi_0)} = i(\tilde{u}(\xi_0))\Omega$ and where we have chosen a basis $\{e_0, e_{0'}, e_1, \ldots, e_{2n}\}$ of the symplectic vector space R^{2n+2} relative to which the symplectic form has matrix

$$\Omega' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \Omega \end{pmatrix} \qquad \Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Consider the local reduction procedure described in section 3 from the element A defined above around the point $x_0 = e_0 \in \Sigma_A = \{x \in \mathbb{R}^{2n+2} \mid \Omega'(x, Ax) = 1\}$.

From what we saw in section 3 this yields a symplectic manifold with a Ricci-type connection (M', ω', ∇').

Denote by π' the map $\pi' : U_{e_0} \to M'$ where U_{e_0} is the neighborhood of e_0 in $\Sigma_A \subset \mathbb{R}^{2n+2}$ considered in section 3 and consider $y_0 = \pi'(e_0)$. Then $\mathscr{H}_{e_0} = \rangle e_0, Ae_0 = e_{0'}\langle^{\perp} = \rangle e_1, \ldots, e_{2n}\langle$ is isomorphic under π'_* to $T_{y_0}M'$.

Introduce the injection $j : \mathbb{R}^{2n} \to \mathbb{R}^{2n+2}$ with $j(x_1, \ldots, x_{2n}) = (0, 0, x_1, \ldots, x_{2n})$ so that $j(\mathbb{R}^{2n}) = \mathscr{H}_{e_0}$ and denote by $b : T_{p_0}M \to T_{y_0}M'$ the map given by

$$b = \pi'_{*_{e_0}} \circ j \circ \xi_0^{-1}.$$

This map b is a linear symplectic isomorphism since

$$\omega'_{y_0}(bX,bY) = \Omega'(j\xi_0^{-1}X, j\xi_0^{-1}Y) = \Omega(\xi_0^{-1}X,\xi_0^{-1}Y) = \omega_{p_0}(X,Y).$$

Furthermore

$$\begin{aligned}
u'(y_0) &= \pi'_{*_{e_0}}(\bar{u'}(x_0)) = \pi'_{*_{e_0}}(-2(n+1)(2n+1)(A^2e_0 - \Omega'(A^2e_0, Ae_0)e_0)) \\
&= \pi'_{*_{e_0}}(j\tilde{u}(\xi_0)) = \pi'_{*_{e_0}}(\xi_0)^{-1}\ u(p_0) = bu(p_0) \\
\rho'(y_0)bX &= \pi'_{*_{e_0}}\overline{\rho'(y_0)X}(e_0) = \pi'_{*_{e_0}}(-2(n+1)\overline{A_{e_0}}(j\xi_0^{-1}(X))) = \pi'_{*_{e_0}}(j\tilde{\rho}(\xi_0)\xi_0^{-1}(X)) \\
&\quad \text{so that } \rho'(y_0)b = b\rho(p_0) \\
(f')(y_0) &= 2(n+1)(2n+1)\Omega'(A^2e_0, Ae_0) = f(p_0).
\end{aligned}$$

Hence we have

Theorem 4.3 *Any real analytic symplectic manifold with a Ricci-type connection is locally symplectically affinely isomorphic to the symplectic manifold with a Ricci-type connection obtained by a local reduction procedure around* $e_0 = (1,0,\dots,0)$ *from a constraint surface* Σ_A *defined by a second order polynomial in the standard flat symplectic manifold* $(\mathbb{R}^{2n+2}, \Omega', \dot{\nabla})$.

5 Construction of a contact manifold which is a global circle or line bundle over M

Consider (M,ω,∇) a smooth symplectic manifold of dimension $2n > 2$ with a smooth Ricci-type connection and let $B(M) \xrightarrow{\pi} M$ be the $Sp(\mathbb{R}^{2n},\Omega)$principal bundle of symplectic frames over M. (An element in the fiber over a point $p \in M$ is a symplectic isomorphism $\xi : (\mathbb{R}^{2n},\Omega) \to (T_pM, \omega_p)$).

As before, we consider $\tilde{u} : B(M) \to \mathbb{R}^{2n}$ the $Sp(\mathbb{R}^{2n},\Omega)$ equivariant function given by

$$\tilde{u}(\xi) = \xi^{-1}u(x) \text{ where } \mathrm{x} = \pi(\xi)$$

and $\tilde{\rho} : B(M) \to sp(\mathbb{R}^{2n},\Omega)$ the $Sp(\mathbb{R}^{2n},\Omega)$ equivariant function given by

$$\tilde{\rho}(\xi) = \xi^{-1}\rho(x)\xi$$

and we define the $Sp(\mathbb{R}^{2n},\Omega)$ equivariant map $\tilde{A}: B(M) \to sp(\mathbb{R}^{2n+2},\Omega')$

$$\tilde{A}(\xi) = \begin{pmatrix} 0 & \dfrac{(\pi^* f)(\xi)}{2(n+1)(2n+1)} & \dfrac{-\underline{\tilde{u}(\xi)}}{2(n+1)(2n+1)} \\ 1 & 0 & 0 \\ 0 & \dfrac{-\tilde{u}(\xi)}{2(n+1)(2n+1)} & \dfrac{-\tilde{\rho}(\xi)}{2(n+1)} \end{pmatrix} \tag{13}$$

where $\underline{V} = i(V)\Omega$ for V in $\mathbb{R}^{2n}$.

We inject the symplectic group $Sp(\mathbb{R}^{2n},\Omega)$ into $Sp(\mathbb{R}^{2n+2},\Omega')$ as the set of matrices

$$\tilde{j}(A) = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix} \qquad A \in Sp(\mathbb{R}^{2n},\Omega).$$

Lemma 5.1 *Define the 1-form α on $B(M)$, with values in $sp(\mathbb{R}^{2n+2},\Omega')$ by:*

$$\alpha_\xi(\overline{X}^{hor}) = \begin{pmatrix} 0 & \dfrac{-\omega_x(u,X)}{2(n+1)(2n+1)} & \dfrac{-\underline{\widetilde{\rho(X)}(\xi)}}{2(n+1)} \\ 0 & 0 & -\underline{\tilde{X}(\xi)} \\ \tilde{X}(\xi) & \dfrac{-\widetilde{\rho(X)}(\xi)}{2(n+1)} & 0 \end{pmatrix} \tag{14}$$

where $X \in T_xM$ with $x = \pi(\xi)$ and $\overline{X}^{hor}$ is the horizontal lift of X in $T_\xi B(M)$, and by:

$$\alpha(C^*) = \tilde{j}_*(C) \tag{15}$$

for all $C \in sp(\mathbb{R}^{2n},\Omega)$ where C^* denotes the fundamental vertical vector field on $B(M)$ associated to C ($C^*_\xi = \frac{d}{dt}\xi.\exp tC_{|0}$).

This form has the following properties:

(i) $R_h^*\alpha = \mathrm{Ad}(\tilde{j}(h^{-1}))\ \alpha \qquad \forall h \in Sp(\mathbb{R}^{2n},\Omega)$;

(ii) $d\tilde{A} = -[\alpha,\tilde{A}]$;

(iii) $d\alpha + [\alpha,\alpha] = -2\tilde{A}\pi^*\omega$

When one has a G-principal bundle $P \xrightarrow{p} M$, an embedding of the group G in a larger group G', $j: G \to G'$, and a 1-form α with values in the Lie algebra of G', such that $\alpha(C^*) = j_*(C)$ for all C in the Lie algebra of G and $R_h^*\alpha = \mathrm{Ad}(j(h^{-1}))\ \alpha$ for all h in G, one can build the $G'-$principal bundle $P' = P \times_G G' \xrightarrow{p'} M$ and the unique connection $1-$form on P', α' satisfying $i^*\alpha' = \alpha$ where $i: P \to P'; \xi \to [(\xi,1)]$.

In our situation we build the $Sp(\mathbb{R}^{2n+2},\Omega')$- principal bundle

$$B'(M)=B(M)\times_{Sp(\mathbb{R}^{2n},\Omega)}Sp(\mathbb{R}^{2n+2},\Omega')$$

whose elements are equivalence classes of pairs (ξ,g) $\xi\in B(M), g\in Sp(\mathbb{R}^{2n+2},\Omega')$ with (ξ,g) equivalent to $(\xi h,\tilde{j}(h^{-1})g)$ $\forall h\in Sp(\mathbb{R}^{2n},\Omega)$.
The projection $\pi':B(M)'\to M$ maps $[(\xi,g)]$ to $\pi(\xi)$.

The connection $1-$form α' is characterised by the fact that

$$\alpha'_{[\xi,1]}([\overline{X}^{hor},0])=\alpha_\xi(\overline{X}^{hor})$$

and the equations above give:

Lemma 5.2 *The curvature $2-$form of the connection $1-$form α' is equal to $-2\tilde{A}'\pi'^*\omega$ where $\tilde{A}'$ is the unique $Sp(\mathbb{R}^{2n+2},\Omega')-$equivariant extension of $\tilde{A}$ to $B'(M)$.*

This curvature $2-$form is invariant by parallel transport ($d^{\alpha'}curv(\alpha')=0$).

Thus the holonomy algebra of α' is of dimension 1.

Corollary 5.3 *Assume M is simply connected. The holonomy bundle of α' is a circle or a line bundle over M, $N\xrightarrow{\pi'}M$. This bundle has a natural contact structure ν given by the restriction to $N\subset B(M)'$ of the $1-$form $-\alpha'$ (viewed as real valued since it is valued in a $1-$dimensional algebra). One has $d\nu=2\pi'^*\omega$.*

It is enlightening to point out the link between the holonomy bundle N over M and the constraint surface Σ_A when one sees M as obtained (locally) by reduction. The link is only local since Σ_A is in general not a principal bundle over M; in fact in most cases the quotient of Σ_A by the action of the group $\exp tA$ is at best an orbifold.

Let A be a nonzero element of $sp(\mathbb{R}^{2n+2},\Omega')$ and let $\Sigma_A=\{y\in\mathbb{R}^{2n+2}\mid\Omega'(y,Ay)=1\}$; we assume as before that it is not empty. Assume that (M,ω,∇) is obtained by reduction from Σ_A (as before, we restrict ourselves to some open set in Σ_A).

Let y_0 be a point in Σ_A, let $x_0=\pi(y_0)\in M$ and choose a symplectic frame ξ_0 at x_0. Let $\gamma(t)$ be a curve in M such that $\gamma(0)=x_0$. Let $\xi(t)$ be the symplectic frame at $\gamma(t)$ obtained by parallel transport along γ from ξ_0 and let $y(t)$ be the horizontal curve in Σ_A lifting $\gamma(t)$ from y_0 (i.e. $\pi(y(t)=\gamma(t)$ and $\Omega'(y(t),\dot{y}(t))=0$). Define the element $C(t)$ of $Sp(\mathbb{R}^{2n+2},\Omega')$ as the matrix whose columns are

$$C(t)=\Big(y(t)\ Ay(t)\ \overline{\xi(t)}\Big)$$

where $\overline{\xi(t)}$ consists of the $2n$ vectors which are the horizontal lifts at the point $y(t)$ of the vectors of the frame $\xi(t)$ (the image under the map $\xi(t)$ of the usual basis of $\mathbb{R}^{2n}$).

Then

$$\frac{d}{dt}C(t)_{|s} = C(s).\alpha_{\xi(s)}(\overline{\dot{\gamma}(s)}^{hor})$$

where α is the 1-form on $B(M)$ defined in (14) and where $\overline{X}^{hor}$ is the horizontal lift of X in $B(M)$; hence $\overline{\dot{\gamma}(s)}^{hor} = \dot{\xi}(s)$.

Let $B'(M)$ be the $Sp(\mathbb{R}^{2n+2}, \Omega')$-principal bundle over M considered above and let $[(\xi_0, \Lambda_0)]$ (where Λ_0 is an element in $Sp(\mathbb{R}^{2n+2}, \Omega')$) be a point of $B'(M)$ above x_0. The horizontal lift of $\gamma(t)$ to $B'(M)$ starting from $[(\xi_0, \Lambda_0)]$ lives in the holonomy subbundle containing this point; it reads

$$[(\xi(t), D(t))]$$

where $\xi(t)$ has been defined above and where $D(t)$ obeys the differential equation

$$\frac{d}{dt}D(t)_{|s} = -\alpha_{\xi(s)}(\dot{\xi}(s)).D(s)$$

and has initial value Λ_0.

Define the map above γ which sends $y(t)$ to $[(\xi(t), D(t))]$ where

$$D(t) = C^{-1}(t)C(0)\Lambda_0;$$

this map sends elements of Σ_A to elements in the holonomy bundle through $[(\xi_0, \Lambda_0)]$.

The map from the holonomy bundle through $[(\xi_0, \Lambda_0)]$ to $\mathbb{R}^{2n+2}$ given by:

$$[(\xi, D)] \mapsto C_0\Lambda_0 D^{-1}e_0$$

where C_0 is a fixed element in $Sp(\mathbb{R}^{2n+2}, \Omega')$ has value in the hypersurface $\Sigma_{A'}$ where $A' = C_0\tilde{A}(\xi_0)C_0^{-1}$.

6 Embedding of the contact manifold in a flat symplectic manifold

Let (M, ω) be a smooth symplectic manifold of dim $2n$ ($n \geq 2$) and let ∇ be a smooth symplectic connection of Ricci-type. Let (N, α) be a smooth $(2n+1)$-dimensional contact manifold (i.e. α is a smooth 1-form such that $\alpha \wedge (d\alpha)^n \neq 0$ everywhere). Let X be the corresponding Reeb vector field (i.e. $i(X)d\alpha = 0$ and $\alpha(X) = 1$). Assume there exists a smooth submersion $\pi : N \to M$ such that $d\alpha = 2\pi^*\omega$. Then at each point $x \in N$, $\operatorname{Ker}(\pi_{*x}) = \mathbb{R}X$ and $\mathscr{L}_X\alpha = 0$.

Remark that such a contact manifold exists always if M is simply connected as we saw in the previous section.

If U is a vector field on M we can define its "horizontal lift" $\overline{U}$ on N by:

$$(i)\ \ \pi_*\overline{U}=0 \qquad (ii)\ \ \alpha(\overline{U})=0.$$

Let us denote by ν the 2-form $\nu=d\alpha=2\pi^*\omega$ on N. Define a connection ∇^N on N by:

$$\begin{aligned}
{\nabla^N}_{\overline{U}}\overline{V} &= \overline{\nabla_U V}-\nu(\overline{U},\overline{V})X\\
{\nabla^N}_X\overline{U} &= {\nabla^N}_{\overline{U}}X=-\frac{1}{2(n+1)}\overline{\rho U}\\
{\nabla^N}_X X &= -\frac{1}{2(n+1)(2n+1)}\overline{u}
\end{aligned}$$

where ρ is the Ricci endomorphism of (M,∇) and where u is the vector field on M appearing in $\nabla\rho$, see lemma 2.1. Then ∇^N is a torsion free connection on N and the Reeb vector field X is an affine vector field for this connection.

The curvature of this connection has the following form:

$$\begin{aligned}
R^N(\overline{U},\overline{V})\overline{W} &= \frac{1}{2(n+1)}[\nu(\overline{\rho V},\overline{W})\overline{U}-\nu(\overline{\rho U},\overline{W})\overline{V}]\\
R^N(\overline{U},\overline{V})X &= \frac{1}{2(n+1)(2n+1)}[\nu(\overline{u},\overline{V})\overline{U}-\nu(\overline{u},\overline{U})\overline{V}]\\
R^N(\overline{U},X)\overline{V} &= \frac{1}{2(n+1)(2n+1)}\nu(\overline{u},\overline{V})\overline{U}+\frac{1}{2(n+1)}\nu(\overline{U},\overline{\rho V})X\\
R^N(\overline{U},X)X &= \frac{1}{2(n+1)(2n+1)}[-\pi^*f\ \overline{U}+\nu(\overline{U},\overline{u})X]
\end{aligned}$$

where f is the function appearing in lemma 2.1.

Consider now the embedding of the contact manifold N into the symplectic manifold (P,μ) of dimension $2n+2$, where

$$P=N\times\mathbb{R}$$

and, if we denote by s the variable along $\mathbb{R}$ and let $\theta=e^{2s}\ p_1^*\alpha$ $(p_1:P\to N)$, we set

$$\mu=d\theta=2e^{2s}\ ds\wedge\alpha+e^{2s}\ d\alpha$$

and let $i:N\to P\ \ x\mapsto(x,0)$. Obviously $i^*\mu=\nu$.

We now define a connection ∇^1 on P as follows. If Z is a vector field along N, we denote by the same letter the vector field on P such that

$$(i)\ \ Z_{i(x)}=i_{*x}Z \qquad (ii)\ \ [Z,\partial_s]=0.$$

The formulas for ∇^1 are:

$$\nabla^1_Z Z' = \nabla^N{}_Z Z' + \gamma(Z, Z')\partial_s$$

where

$$\begin{aligned}
\gamma(Z, Z') &= \gamma(Z', Z) \\
\gamma(X, X) &= \frac{1}{2(n+1)(2n+1)}\pi^* f \\
\gamma(X, \overline{U}) &= -\frac{1}{2(n+1)(2n+1)}\nu(\overline{u}, \overline{U}) \\
\gamma(\overline{U}, \overline{V}) &= \frac{1}{2(n+1)}\nu(\overline{U}, \overline{\rho V})
\end{aligned}$$

and

$$\begin{aligned}
\nabla^1_Z \partial_s &= \nabla^1_{\partial_s} Z = Z \\
\nabla^1_{\partial_s}\partial_s &= \partial_s.
\end{aligned}$$

Theorem 6.1 *The connection ∇^1 on (P, μ) is symplectic and has zero curvature.*

Proposition 6.2 *Let $\psi(s)$ be a smooth function on P. Then ψ has vanishing third covariant differential if and only if*

$$\partial_s^2\psi - 2\partial_s\psi = 0. \tag{16}$$

In particular the function e^{2s} has this property.

The procedure described above is called the induction.

Let (P, μ, ∇^1) be as above and let $\Sigma = N$ be the constrained submanifold defined by $e^{2s} = 1$. Let Y be the vector field transversal to Σ such that $i(Y)\mu = \alpha$, thus $Y = \partial_s$.

Let H be the 1-parametric group generated by X. Then Σ/H can be identified with M and (M, ω) is the classical Marsden Weinstein reduction of (P, μ) for the constraint Σ.

The connection ∇ on M is obtained from the flat connection ∇^1 on (P, μ) by reduction. Hence

Corollary 6.3 *Any smooth simply connected symplectic manifold with a Ricci-type connection (M, ω, ∇) can be obtained by reduction from an hypersurface Σ in a flat symplectic manifold (P, μ, ∇^1) defined by the $1-$level set of a function ψ on P whose third covariant derivative vanishes.*

Corollary 6.4 *Any smooth simply connected symplectic manifold with a Ricci-type connection (M, ω, ∇) is automatically analytic.*

Proof Since (P, μ, ∇^1) is locally symmetric, P, μ and ∇^1 are real analytic and the explicit construction given preserves analyticity. $\square$

7 Symmetric symplectic spaces with Ricci-type connections

Lemma 7.1 *The reduction construction described in section 3 yields a locally symmetric symplectic space (i.e. such that the curvature tensor is parallel) if and only if the element $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ satisfies $A^2 = \lambda I$ for a constant $\lambda \in \mathbb{R}$.*

Proof Indeed the connection ∇ has parallel curvature tensor if and only if $\nabla\rho = 0$ hence iff $u = 0$. From the formulas above, this is true iff

$$\overline{A^2_x}(x) = A^2x - \Omega'(A^2x, Ax)x = 0$$

for any $x \in \Sigma_A$. When $u = 0$, f is a constant (cf Lemma 2.1) and it follows from Lemma 3.1 that $\Omega'(A^2x, Ax)$ is a constant λ. Since Σ_A contains a basis of $\mathbb{R}^{2n+2}$, this yields $A^2 = \lambda I$. $\square$

Proposition 7.2 *If $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ satisfies $A^2 = \lambda I$ for a constant $\lambda \in \mathbb{R}$, the quotient of Σ_A by the action of $\exp tA$ is a manifold and the natural projection map $\Sigma_A \to M$ is a submersion which endows Σ_A with a structure of circle or line bundle over M.*

Proof Consider $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ so that $A^2 = \lambda I$.

-Case 1: $\lambda > 0$, say $\lambda = k^2$ with $k > 0$.

Then there exists a basis of $\mathbb{R}^{2n+2}$ in which

$$A = \begin{pmatrix} kI_{n+1} & 0 \\ 0 & -kI_{n+1} \end{pmatrix} \qquad \Omega' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

so that $\Sigma_A = \{(u, v)\ u, v \in \mathbb{R}^{n+1} \mid -2ku \cdot v = 1\}$. The flow of the vector field Ax is given by $\phi_t = e^{tA}$.

The map $\pi : \Sigma_A \to TS^n = \{(u', v')\ u', v' \in \mathbb{R}^{n+1}\ |u' \cdot u' = 1, u' \cdot v' = 0\}$ defined by

$$\pi(u, v) = \left(\frac{u}{\|u\|}, \|u\|(v + \frac{u}{2k\|u\|^2}) \right)$$

induces a diffeomorphism between $M = \Sigma_A/_{\phi_t}$ and TS^n.

M is a non compact simply connected manifold and Σ_A is a $\mathbb{R}-$bundle over TS^n.

-Case 2 : $\lambda < 0$, say $\lambda = -k^2$ with $k > 0$.

One splits $V^{\mathbb{C}}$ ($V = \mathbb{R}^{2n+2}$) into the eigenspaces relative to A, $V^{\mathbb{C}} = V_{ik} \oplus V_{-ik}$ and observe that those subspaces are Lagrangian. Choosing a basis $\{z_1, \ldots, z_{n+1}\}$ for V_{ik}, consider $\omega_{kl} := \Omega'(z_k, \overline{z}_l)$; then $i\omega$ is a Hermitian matrix. A change of basis ($z'_j = \sum z_i U^i_j$) yields $\omega' =^t U\omega U$ so we can find a basis for V_{ik} so that $\omega = -2iI_{p,n+1-p} = -2i\begin{pmatrix} I_p & 0 \\ 0 & -I_{n+1-p} \end{pmatrix}$. In the basis of $\mathbb{R}^{2n+2}$ given by $e_j = \frac{1}{2}(z_j + \overline{z}_j)$ $\quad f_j = \frac{1}{2i}(z_j - \overline{z}_j)$ we have:

$$A = \begin{pmatrix} 0 & -kI \\ kI & 0 \end{pmatrix} \qquad \Omega' = \begin{pmatrix} 0 & I_{p,n+1-p} \\ -I_{p,n+1-p} & 0 \end{pmatrix}$$

so that $\Sigma_A = \{(u,v)\ u,v \in \mathbb{R}^{n+1} \mid k\sum_{i\leq p}((u^i)^2 + (v^i)^2) - k\sum_{i>p}((u^i)^2 + (v^i)^2) = 1\}$. We assume $p \geq 1$ or replace A by $-A$ so that $\Sigma_A \tilde{=} S^{2p-1} \times \mathbb{R}^{2n-2p+2}$ is non empty. The flow ϕ_t is given by the action of $\exp tA = \begin{pmatrix} \cos ktI & -\sin ktI \\ \sin ktI & \cos ktI \end{pmatrix}$.

Then $M = \Sigma_A/_{\phi_t} = (S^{2p-1} \times \mathbb{R}^{2n-2p+2})/U(1)$, so this reduced manifold is:

- $M = \mathbb{R}^{2n}$ if $p = 1$;
- M is a complex line bundle of rank $q := n + 1 - p$ over the complex projective space $P_{p-1}(\mathbb{C}) = S^{2p-1}/U(1)$ if $1 < p \leq n$;
- $M = P_n(\mathbb{C})$ if $p = n + 1$.

In all those cases, M is simply connected and Σ_A is a circle bundle over M; the only compact case is $M = P_n(\mathbb{C})$.

-Case 3 : $\lambda = 0$, so $A^2 = 0$ with $A \neq 0$. Let us denote by p the rank of A. One splits $V = \mathbb{R}^{2n+2}$ into $V = V_0 \oplus V_1 \oplus V_2$ where $V_1 = ImA$ (dim $V_1 = p$), $V_0 \oplus V_1 = \text{Ker } A$ (so dim $V_0 = 2n + 2 - 2p$ and V_0 is symplectic, since $V_0 \oplus V_1 = V_1^{\perp}$) and V_2 is a Lagrangian subspace of $V_0^{\perp}$ supplementary to V_1. Choose a basis of V_2 and a corresponding basis (dual for Ω') in V_1 and a symplectic basis of V_0 so that in those basis

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A' \\ 0 & 0 & 0 \end{pmatrix} \qquad \Omega' = \begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & 0 & I_p \\ 0 & -I_p & 0 \end{pmatrix}$$

and A' is symmetric. Changing the basis of V_2 and correspondingly the basis of V_1, one can bring A' to the form $A' = I_{r,p-r}$ so that $\Omega'(x, Ax) = \sum_{i\leq r}(w^i)^2 - \sum_{r<i\leq p}(w^i)^2$ if $x = (u, v, w)$.

Hence $\Sigma_A = S^{r-1} \times \mathbb{R}^{2n+2-r}$ if $r > 1$ and Σ_A consists of two copies of $\mathbb{R}^{2n+1}$ if $r = 1$.

The action of ϕ_t on (u,v,w) is given by $\phi_t(u,v,w)=(u,v+tA'w,w)$ so the reduced manifold is

-two copies of $\mathbb{R}^{2n}$ (if $r=1$);

-or $M=S^{r-1}\times\mathbb{R}^{2n+1-r}$ if $r>1$.

In all cases, M is a non compact manifold and Σ_A is a line bundle over M. $\square$

Proposition 7.3 *If $0\neq A\in sp(\mathbb{R}^{2n+2},\Omega')$ satisfies $A^2=\lambda I$ for a constant $\lambda\in\mathbb{R}$, the quotient manifold is a symmetric space and the connection obtained by reduction is the canonical symmetric connection.*

Proof Any linear symplectic transformation B of $\mathbb{R}^{2n+2}$ which commutes with A obviously induces a symplectic affine transformation $\beta(B)$ of the reduced space $M=\Sigma_A/\phi_t$. If π denotes the canonical projection $\pi:\Sigma_A\to M$, then

$$\beta(B)\circ\pi=\pi\circ B.$$

In particular the symmetry at the point $x=\pi(y), y\in\Sigma_A$ is induced by

$$B_y u=-u+2\Omega'(u,Ay)y-2\Omega'(u,y)Ay.$$

$\square$

We shall now describe the tranvection group of M (i.e. the group G of affine transformations of M generated by the composition of two symmetries). Let us denote by G' the group $G'=\{B\in Sp(R^{2n+2},\Omega')\mid BA=AB\}$. The tranvection group of M is clearly included in $\beta(G')$; in fact it is the smallest subgroup of $\beta(G')$ stable under conjugation by a symmetry and which acts transitively on M.

Let $x_0=\pi(y_0)$ be a point in M and let $s_{x_0}=\beta(B_{y_0})$ be the symmetry at this point. Consider the automorphism of G' given by conjugaison by $B_{y_0}$and denote by σ the induced automorphism of the Lie algebra $\mathfrak{g}'$ of G'. Let $\mathfrak{p}'=\{C\in\mathfrak{g}'\mid\sigma(C)=-C\}$ and $\mathfrak{k}'=\{C\in\mathfrak{g}'\mid\sigma(C)=C\}$.

The dimension of $\mathfrak{p}'$ is equal to $2n$. Indeed, in a basis $\{e_0,e_{0'},e_1,\ldots,e_{2n}\}$ of R^{2n+2} in which $e_0=y_0$ and $e_{0'}=Ae_0$ and $\Omega'=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&\Omega\end{pmatrix}$, one has

$$B_{y_0}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-I_{2n}\end{pmatrix}\qquad A=\begin{pmatrix}0&\lambda&0\\1&0&0\\0&0&A'\end{pmatrix},$$

$$\mathfrak{g}' = \{ \begin{pmatrix} b & \lambda c & \underline{A'Z} \\ c & -b & -\underline{Z} \\ Z & A'Z & B \end{pmatrix} \quad b, c \in \mathbb{R}; Z \in \mathbb{R}^{2n}; B \in sp(\mathbb{R}^{2n}, \Omega) \text{ such that } \mathrm{BA}' = \mathrm{A}'\mathrm{B}\},$$

and $\mathfrak{p}' = \{ \begin{pmatrix} 0 & 0 & \underline{A'Z} \\ 0 & 0 & -\underline{Z} \\ Z & A'Z & 0 \end{pmatrix} \quad Z \in \mathbb{R}^{2n}\}$.

Hence the Lie algebra of the transvection group is equal to $\beta_*(\mathfrak{p}' + [\mathfrak{p}', \mathfrak{p}'])$.
In all cases the kernel of β is given by $\exp tA$, and the transvection group is described as follows:

-Case 1: $\lambda > 0$, say $\lambda = k^2$ with $k > 0$.

In the basis of $\mathbb{R}^{2n+2}$ in which $A = \begin{pmatrix} kI_{n+1} & 0 \\ 0 & -kI_{n+1} \end{pmatrix} \quad \Omega' = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$,
we have $G' = \{ \begin{pmatrix} B & 0 \\ 0 & ({}^tB)^{-1} \end{pmatrix} \quad B \in Gl(n+1, \mathbb{R})\}$, and β of such an element is the identity iff $B = \lambda I$ with $\lambda > 0$.

The transvection group G is isomorphic to $Sl(n+1, \mathbb{R})$ and

$$TS^n = Sl(n+1, \mathbb{R})/Gl(n, \mathbb{R}).$$

-Case 2 : $\lambda < 0$, say $\lambda = -k^2$ with $k > 0$.

In the basis of $\mathbb{R}^{2n+2}$ in which $A = \begin{pmatrix} 0 & -kI \\ kI & 0 \end{pmatrix} \quad \Omega' = \begin{pmatrix} 0 & I_{p,n+1-p} \\ -I_{p,n+1-p} & 0 \end{pmatrix}$
we have $G' = \{ \begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix} \quad B_1 + iB_2 \in U(p, n+1-p)\}$, and β of such an element is the identity iff $B_1 + iB_2 = \exp -ikt$.

The transvection group G is isomorphic to $SU(p, n+1-p)$ and

$$M = SU(p, n+1-p)/U(p-1, n+1-p).$$

-Case 3: $\lambda = 0$, $\mathrm{rank} A = k = p + q$.

In the basis of $\mathbb{R}^{2n+2}$ in which $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{pq} \\ 0 & 0 & 0 \end{pmatrix}$ and $\Omega' = \begin{pmatrix} \Omega_1 & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & 0 \end{pmatrix}$
we have $\mathfrak{g}' = \{ \begin{pmatrix} D & 0 & C \\ -{}^tC\Omega_1 & -{}^tB & F \\ 0 & 0 & B \end{pmatrix} \quad D \in sp(\mathbb{R}^{2n+2-2k}, \omega_1), B \in so(p, q, \mathbb{R}), F \in gl(k, \mathbb{R}), {}^tF = F, C \in Mat(2n+2-2k, k, \mathbb{R})\}$.

Then $\mathfrak{p}'$ is given by the elements of $\mathfrak{g}'$ for which $D = 0, C = CJ, F = -JFJ, B = -JBJ$ where $J = \begin{pmatrix} 1 & 0 \\ 0 & -I_{k-1} \end{pmatrix}$, so $C = (u\,0\,\ldots\,0)$ for $u \in \mathbb{R}^{2n+2-2k}$, $F = \begin{pmatrix} 0 & {}^t v \\ v & 0 \end{pmatrix}$ for $v \in \mathbb{R}^{k-1}$ and $B = \begin{pmatrix} 0 & {}^t w' \\ w & 0 \end{pmatrix}$ for $w \in \mathbb{R}^{k-1}$ and $w' = I_{p-1,q}w$.

Hence $\mathfrak{p}' \oplus [\mathfrak{p}', \mathfrak{p}']$ is the set of all elements in $\mathfrak{g}'$ for which $D = 0$.

The transvection group G has algebra $\mathfrak{g}$ isomorphic to $\{(B, F, C)\}/(0, \mathbb{R}I_{pq}, 0)$ where B is any element in $so(p, q, \mathbb{R})$, F is any symmetric real $k \times k$ matrix , and C is any real $(2n + 2 - 2k) \times k$ matrix and the bracket is defined by

$$[(B, F, C), (B', F', C')] = ([B, B'], -{}^tC\Omega_1 C' +^t C'\Omega_1 C -^t BF' +^t B'F, CB' - C'B),$$

so when $p + q > 2$, the Levi factor is $so(p, q, \mathbb{R})$ and the radical is a 2−step nilpotent algebra. If $p = 0$ and $q = 1$ the transvection group is $\mathbb{R}^{2n}$ and the symmetric space is the standard symplectic vector space. If $p = q = 1$ or if $p = 0$ and $q = 2$, the transvection group is solvable but not nilpotent. The two solvable examples are interesting for building exact quantisation.

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